FLUID LEAKAGE FROM NONUNIFORMLY PERMEABLE POROUS RESERVOIR THROUGH A RELATIVELY IMPERMEABLE BARRIER

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We examine fluid leakage in the case of forced filtration in a porous reservoir of nonuniform permeability which is separated by a low-permeability barrier from an overlying relatively impermeable reservoir; the horizontal component of the filtration velocity in the barrier is neglected (scheme of Shchelkachev and Hussein-zade). The permeability of the porous reservoir is approximated by a finite continuous function

$$k = k(z) \tag{0.1}$$

of the vertical coordinate z.

An analytic solution of the problem is constructed in the plane and axisymmetric cases in the form of series in regular functions which are easily tabulated for concretely specified permeability. A numerical example is presented for two cases of specification of the permeability k(z).

1. Assume fluid is pumped through a straight gallery of injection wells into an isotropic porous reservoir with the indicated nonuniformity. We assume the pressure above the barrier is constant, the reservoir floor is impermeable, and the forced filtration is steady-state, obeying the linear Darcy law. Then the problem will be described by the equation [1]

$$\frac{\partial}{\partial x}\left(k(z)\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial z}\left(k(z)\frac{\partial p}{\partial z}\right) = 0 \tag{1.1}$$

and the boundary conditions

$$p = p^*, \; x = 0; \qquad p = p^\circ, \quad x = L; \qquad \frac{\partial p}{\partial z} = 0, \quad z = 0$$
$$\frac{\partial p}{\partial z} + \frac{k_1}{hk(H)}(p - p_0) = 0, \qquad z = H \tag{1.2}$$

The notations are: p = reservoir pressure; p* = pressure along injection gallery; p° = pressure at edge x = L of gallery influence zone; k_1 = barrier permeability; h = barrier thickness; H = reservoir thickness; p_0 = pressure above barrier.

We write (1.1) in the form

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial z^2} + N(z)\frac{\partial p}{\partial z} = 0, \qquad N(z) = \frac{d}{dz}\ln k(z) \tag{1.3}$$

and seek the solution of this equation with the aid of the series [2]

$$p = p^* - \frac{p^* - p^\circ}{L}x + \sum_{m=0}^{\infty} G_m(x, z)\, g_m(z) \tag{1.4}$$

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where limitations of the form

$$\frac{\partial^2 G_m}{\partial x^2}+\frac{\partial^2 G_m}{\partial z^2}=0 \quad (m=0,1,2,\ldots), \qquad \frac{\partial G_m}{\partial z}=G_{m-1} \quad (m=1,2,\ldots) \tag{1.5}$$

are imposed on the functions $G_m(x, z)$.

With account for these conditions the series (1.4) will be the solution of (1.3) if the following differential recursion relations are satisfied for the functions $g_m(z)$:

$$2g'_0+Ng_0=0, \quad 2g_m'+Ng_m=-(g''_{m-1}+Ng_{m-1}') \quad (m=1,2,\ldots) \tag{1.6}$$

In finding $g_m(z)$ we restrict ourselves to the particular solutions of (1.6), i.e., we select the class of particular solutions of (1.3) which depend only on the single harmonic function $G_0(x, z)$; all the subsequent functions $G_m(x, z)$ are expressed in terms of the preceding functions and in the final analysis in terms of $G_0(x, z)$.

Solving (1.6), we find

$$g_0=k^{-1/2}(z), \quad g_m=-\frac{1}{2\sqrt{k(z)}}\int k^{-1/2}(z)\,(k(z)\,g_{m-1}')'\,dz \quad (m=1,2,\ldots) \tag{1.7}$$

The integrals (1.7) can be tabulated for concretely specified permeability of the porous reservoir.

Considering that the functions $G_m(x, z)$ must satisfy zero boundary conditions at the vertical boundaries of the filtration region, we take the harmonic function $G_0(x, z)$ in the form

$$G_0(x,z)=\sum_{n=1}^{\infty}(a_n e^{\lambda_n z}+b_n e^{-\lambda_n z})\sin\lambda_n x, \quad \lambda_n=n\pi/L \tag{1.8}$$

From (1.5) we obtain for $G_m(x, z)$

$$G_m(x,z)=\sum_{n=1}^{\infty}(a_n e^{\lambda_n z}+(-1)^m b_n e^{-\lambda_n z})\lambda_n^{-m}\sin\lambda_n x \quad (m=1,2,\ldots) \tag{1.9}$$

We introduce the notations

$$\alpha_n(z)=e^{\lambda_n z}\sum_{m=0}^{\infty}\lambda_n^{-m}g_m(z), \quad \beta_n(z)=e^{-\lambda_n}\sum_{m=0}^{\infty}(-1)^m\lambda_n^{-m}g_m(z) \tag{1.10}$$

In these notations (1.4) can be written as

$$p=p^*-\frac{p^*-p^\circ}{L}x+\sum_{n=1}^{\infty}(a_n\alpha_n(z)+b_n\beta_n(z))\sin\lambda_n x \tag{1.11}$$

Subjecting (1.11) to the boundary conditions at the roof and floor of the reservoir, we obtain the following system of algebraic equations in the coefficients a_n and b_n:

$$a_n\alpha_n'(0)+b_n\beta_n'(0)=0, \quad a_n\alpha_n'(H)+b_n\beta_n'(H)+\alpha(a_n\alpha_n(H)+b_n\beta_n(H))=c_n \tag{1.12}$$

$$c_n=\frac{2\alpha}{\lambda_n L}[(p^*-p_0)+(-1)^{n-1}(p^\circ-p_0)], \quad \alpha=\frac{k_1}{hk(H)} \tag{1.13}$$

From (1.2) we find

$$a_n=\frac{\beta_n'(0)c_n}{\alpha_n'(0)(\beta_n'(H)-\alpha\beta_n(H))-\beta_n'(0)(\alpha_n'(H)+\alpha\alpha_n(H))}, \quad b_n=\frac{\alpha_n'(0)c_n a_n}{\beta_n'(0)} \tag{1.14}$$

The injection gallery flowrate, flowrate at the contour $x = L$, and the relative leakage per unit reservoir width can now be defined by the equalities

$$Q_0=\int_0^H k(z)\left(\frac{\partial p}{\partial x}\right)_{x=0}dz, \quad Q_L=\int_0^H k(z)\left(\frac{\partial p}{\partial x}\right)_{x=L}dz, \quad \eta=\frac{Q_0-Q_L}{Q_0} \tag{1.15}$$

For two particular cases of permeability specified in the form of the functions

$$k^{(1)}(z)=k_0(1+0.2z)^2, \quad k^{(2)}(z)=k_0(5-0.2z)^2$$

the numerical calculation was made using the initial values

$$k_0 = 30\ \text{ppm},\ k_1 = 0.001\ \text{ppm},\ h = 10\,\text{m},\ H = 20\,\text{m},\ L = 5000\ \text{m}$$
$$p^* = 180\ \text{atm},\quad p^\circ = p_0 = 100\ \text{atm}$$

For the relative leakage in the first and second cases, respectively, the calculations showed that $\eta^{(1)} = 0.073$ and $\eta^{(2)} = 0.251$, while in the case of the exact average permeability (the same in both cases) $\eta = 0.144$.

2. In the case of axisymmetric fluid flow from a central injection well, the reservoir pressure will be defined by the following boundary value problem

$$\frac{\partial^2 p}{\partial z^2} + \frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + N(z)\frac{\partial p}{\partial z} = 0 \tag{2.1}$$

$$p = p^*\ (r = r_0),\qquad p = p^\circ\ (r = R),\qquad \frac{\partial p}{\partial z} = 0\quad (z = 0)$$
$$\frac{\partial p}{\partial z} + \frac{k_1}{hk(H)}(p - p_0) = 0\ (z = H) \tag{2.2}$$

Here r_0 and R are the radii of the well and well influence zone; and r and z are cylindrical coordinates.

We write the solution of (2.1) in series form

$$p = p^\circ + \frac{p^* - p^\circ}{\ln(r_0/R)}\ln\frac{r}{R} + \sum_{m=0}^{\infty} F_m(r, z) f_m(z) \tag{2.3}$$

where $F_m(r, z)$ satisfy the conditions

$$\frac{\partial^2 F_m}{\partial z^2} + \frac{\partial^2 F_m}{\partial r^2} + \frac{1}{r}\frac{\partial F_m}{\partial r} = 0\quad (m = 0, 1, 2, \ldots),\qquad \frac{\partial F_m}{\partial z} = F_{m-1}\quad (m = 1, 2, \ldots) \tag{2.4}$$

and the recursion relation (1.7) holds for the functions $f_m(z)$.

We take the function $F_0(r, z)$ in the form

$$F_0(r, z) = \sum_{n=1}^{\infty} (a_n e^{\lambda_n z} + b_n e^{-\lambda_n z}) U_n(\lambda_n r) \tag{2.5}$$
$$U_n(\lambda_n r) = J_0(\lambda_n r) Y_0(\lambda_n r_0) - J_0(\lambda_n r_0) Y_0(\lambda_n r) \tag{2.6}$$

where $J_0(\lambda_n r)$ and $Y_0(\lambda_n r)$ are Bessel functions of first and second kinds, respectively, of a real argument, and λ_n will be roots of the equation

$$J_0(\lambda_n R) Y_0(\lambda_n r_0) - J_0(\lambda_n r_0) Y_0(\lambda_n R) = 0 \tag{2.7}$$

Using (2.4), we obtain the expression for the functions $F_m(r, z)$

$$F_m(r, z) = \sum_{n=1}^{\infty} \lambda_n^{-m}(a_n e^{\lambda_n z} + (-1)^m b_n e^{-\lambda_n z}) U_n(\lambda_n r) \tag{2.8}$$

Considering the notations (1.10), the solution (2.3) takes the form

$$p = p^\circ + \frac{p^* - p^\circ}{\ln(r_0/R)}\ln\frac{r}{R} + \sum_{n=1}^{\infty} (a_n\alpha_n(z) + b_n\beta_n(z)) U_n(\lambda_n r) \tag{2.9}$$

To find the coefficients a_n and b_n we use the boundary conditions at the roof and floor of the reservoir

$$a_n\alpha_n'(0) + b_n\beta_n'(0) = 0,\quad a_n\alpha_n'(H) + b_n\beta_n'(H) + \alpha(a_n\alpha_n(H) + b_n\beta_n(H)) = -c_n \tag{2.10}$$

where c_n are the coefficients of the expansion of the function

$$\psi(r) = \alpha\left(p^\circ - p^* + \frac{p^* - p^\circ}{\ln(r_0/R)}\ln\frac{r}{R}\right)$$

into a series in the orthogonal functions $U_n(\lambda_n r)$.

For the well flowrate, the flowrate at the boundary of the well influence zone, and the relative fluid leakage from the reservoir, we have

$$Q_0 = 2\pi r_0\int_0^H k(z)\left(\frac{\partial p}{\partial r}\right)_{r=r_0} dz,\quad Q_R = 2\pi R\int_0^H k(z)\left(\frac{\partial p}{\partial r}\right)_{r=R} dz,\quad \eta = \frac{Q_0 - Q_R}{Q_0} \tag{2.11}$$

The numerical calculation made for the same permeability values as in the plane problem showed, respectively, $\eta^{(1)}=0.08$, $\eta^{(2)}=0.294$, and $\eta=0.152$. We took the initial values $k_1=0.001$ ppm, $h=10$ m, $H=20$ m, $r_0=0.15$ m, $R=1500$ m, $p^*=180$ atm, $p°=p_0=100$ atm. We see from these examples that averaging of the permeability across the reservoir thickness increases the relative leakage through the relatively impermeable barrier if the permeability k(z) near the reservoir roof is larger than the averaged value, and reduces the leakage if the permeability near the roof is less than the permeability averaged across the reservoir thickness, and the relative error obtained when replacing the true permeability by the averaged value may be very large.

In conclusion, we note that the results of [3] for a uniform reservoir follow from the solutions obtained here if k(z) = const.

The solution can be constructed similarly in the case when the permeability undergoes a discontinuity on N planes $z=z_i$ (i = 1, 2, ..., N) and the porous reservoir has constant anisotropy.

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